An Introduction of Multiple Scales in a Dynamical Cosmology

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Abstract

The discovery of scale acceleration evidenced from supernovae luminosities and spatial flatness of feature evolution in the cosmic microwave background presents a challenge to the understanding of the evolution of cosmological vacuum energy. Although some scenarios prefer a fixed cosmological constant with dynamics governed in a Friedman-Robertson-Walker (FRW) geometry, an early inflationary epoch remains a popular model for cosmology. It is therefore advantageous to develop a metric framework that allows a transition from an early inflationary period to a late stage dominated by dark energy. Such a metric is here developed, and some properties of this metric are explored.

1 Introduction

A cosmological constant is the favored scenario for a quantum cosmology with a single, late time deSitter geometry. For such models, any initial quantum state with an energy density determined in the absence of microscopic particulate excitations (ie the particle vacuum) undergoes a transition to thermal particles, with the cosmological constant generated by the vacuum modes of the initial collective state[1]. The metric of such a system should be described by the standard Friedman-Robertson-Walker (FRW) geometry, with an additional true cosmological constant term in the Einstein equation connecting the geometry to the energy content. However, if there is an early inflation stage, the cosmology should involve a scale transition connecting the early time inflation scale to a later deSitter scale. In such a scenario, a true cosmological constant cannot represent the early evolution of the cosmology, and can at most represent a late time behavior of the cosmology.

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2 Fluid Cosmology

It will be assumed that the dynamics of the cosmology can be accurately described by the Einstein equation during the period under consideration:

$$G_{\mu\nu} \equiv \mathbf{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathbf{R} = - \left(\frac{8\pi G_N}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu} \right). \tag{2.1}$$

For the present discussion, the cosmology evolves in the absence of any true cosmological constant $\Lambda_{true} = 0$. For an ideal fluid (no dissipation), the energy-momentum tensor takes the form

$$T_{\mu\nu} = P\, g_{\mu\nu} + (\rho + P) u_\mu u_\nu, \tag{2.2}$$

where the four velocity of the fluid satisfies the consistency condition

$$u_\mu g^{\mu\nu} u_\nu = -1. \tag{2.3}$$

We assume isotropic flow $u_\theta = 0 = u_\phi$. The trace of the energy-momentum tensor is given by $g^{\mu\nu} T_{\mu\nu} \equiv T^\mu_\mu = 3P - \rho$. This gives the form of the pressure and density in terms of geometric quantities:

$$\begin{array}{l} P = T^\phi_\phi = T^\theta_\theta = -\frac{c^4}{8\pi G_N} G^\theta_\theta, \\ \rho = 3P + \frac{c^4}{8\pi G_N} G^\mu_\mu. \end{array} \tag{2.4}$$

Likewise, the time and radial components of the flow field can be determined to satisfy

$$\begin{array}{l} u_0^2 = \left(T_{00} - g_{00} P \right) / (\rho + P) = - \left(\frac{c^4}{8\pi G_N} G_{00} + g_{00} P \right) / (\rho + P) \\ u_r^2 = \left(T_{rr} - g_{rr} P \right) / (\rho + P) = - \left(\frac{c^4}{8\pi G_N} G_{rr} + g_{rr} P \right) / (\rho + P). \end{array} \tag{2.5}$$

For later comparison, for the Friedman-Robertson-Walker geometry

$$g_{\mu\nu} = -c^2 dt^2 + R^2(ct) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \right), \tag{2.6}$$

these fluid parameters take the form

$$\begin{array}{l} \rho = \frac{3c^4}{8\pi G_N} \left[\frac{\kappa}{R^2} + \left(\frac{\dot{R}}{R} \right)^2 \right], \\ \\ P = -\frac{c^4}{8\pi G_N} \left[\frac{\kappa}{R^2} + \left(\frac{\dot{R}}{R} \right)^2 + 2\frac{\ddot{R}}{R} \right], \\ \\ u_0 = -1 \quad , \quad u_r = 0. \end{array} \tag{2.7}$$

3 Multiple Cosmological Scales

3.1 The river model

The so called "river model" has been explored by some authors[2] to gain insight into the nature of black holes. In general, the metric takes an off diagonal form

$$ds^2 = -dt_R^2 + [dr - \beta(r)dt_R]^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.1}$$

The speed β has been interpreted by some to be the speed of radial outflow of the space-time river through which objects move using the rules of special relativity. A transformation can be developed that diagonalizes the metric. Substitution of the form

$$t_R = t_* - \int^r \frac{\beta(r')}{1-\beta^2(r')} dr' \tag{3.2}$$

gives

$$ds^2 = -(1-\beta^2(r))dt_*^2 + \frac{dr^2}{1-\beta^2(r)} + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.3}$$

The river speed becomes luminal at the horizon associated with the (ct_*, r) coordinates.

Rather than exploring the horizon associated with a black hole, the geometry of deSitter space is of interest for the present discussion. If one explores the geometry given by the metric

$$g_{\mu\nu} = -\left(1 - \frac{r^2}{R_v^2(ct)}\right) c^2 dt^2 - \frac{2\,r}{R_v(ct)} cdt\, dr + \left(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right), \tag{3.4}$$

the fluid parameters generated in Eq. 2.4 are given by

$$\begin{aligned} \rho &= \tfrac{3c^4}{8\pi G_N} \left(\tfrac{1}{R_v}\right)^2, \\ P + \rho &= -\tfrac{c^4}{4\pi G_N} \tfrac{d}{dct} \left(\tfrac{1}{R_v}\right) = \tfrac{c^4}{4\pi G_N} \tfrac{\dot{R}_v}{R_v^2}, \\ u_0 &= -1 \quad , \quad u_r = 0. \end{aligned} \tag{3.5}$$

The fluid is seen to be at rest with respect to the space-time coordinates, with a radial "river" flow speed given by $\beta = r/R_v(ct)$.

A static deSitter geometry can be further transformed into the form of the Friedman-Robertson-Walker geometry by comparing the metric forms

$$
\begin{aligned}
ds^2 &= -(1 - r^2/R_v^2)c^2dt^2 + \tfrac{dr^2}{1-r^2/R_v^2} + r^2(d\theta^2 + sin^2\theta d\phi^2) \\
ds^2 &= -c^2d\tilde{t}^2 + \Delta[d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + sin^2\theta d\phi^2)].
\end{aligned}
\tag{3.6}
$$

The radial scales are seen to satisfy $r = \tilde{r}\Delta$ from angular constraints. A coordinate tranformation can be found for $\frac{\dot{\Delta}}{\Delta} = \frac{1}{R_v}$ and $\dot{R}_v = 0$, giving

$$
\begin{aligned}
&ct = c\tilde{t} - \tfrac{R_v}{2} log \left(1 - \tfrac{\tilde{r}^2}{R_v^2} e^{2c\tilde{t}/R_v}\right), \\
&r = \tilde{r} e^{c\tilde{t}/R_v},
\end{aligned}
\tag{3.7}
$$

or alternatively

$$
\begin{aligned}
&c\tilde{t} = ct + \tfrac{R_v}{2} log \left(1 - \tfrac{r^2}{R_v^2}\right), \\
&\tilde{r} = \tfrac{r e^{-ct/R_v}}{\left(1-\frac{r^2}{R_v^2}\right)^{1/2}}.
\end{aligned}
\tag{3.8}
$$

3.2 Dynamic horizon scale in an isotropic space

Motivated by the river model, one can construct a metric that incorporates the dynamic scales of FRW geometries with asymptotic behaviors similar to deSitter geometries. Consider the following hybrid metric:

$$
\begin{aligned}
g_{\mu\nu} = &-\left(1 - \tfrac{R^2(ct)r^2}{R_v^2(ct)}\right) c^2dt^2 - 2\tfrac{R^2(ct)r}{R_v(ct)} cdt\, dr \\
&+R^2(ct)\left(dr^2 + r^2d\theta^2 + r^2 sin^2\theta d\phi^2\right) \\
&= -c^2dt^2 + R^2(ct)\left(dr - \tfrac{r}{R_v(ct)}cdt\right)^2 \\
&\quad +R^2(ct)\left(r^2d\theta^2 + r^2 sin^2\theta d\phi^2\right).
\end{aligned}
\tag{3.9}
$$

The radial null geodesics satisfy

$$
dr_L = \left(\pm\frac{1}{R} + \frac{r_L}{R_v}\right) cdt, \tag{3.10}
$$

with the particle horizon defined by the values of the scales R, R_v at $t = 0$, with $r_L(0) = 0$ The hydrodynamic parameters can be immediately calculated using Eq. 2.4:

$$
\begin{aligned}
&\rho = \tfrac{3c^4}{8\pi G_N}\left(\tfrac{1}{R_v} + \tfrac{\dot{R}}{R}\right)^2 , \\
&P = -\rho - \tfrac{c^4}{4\pi G_N}\tfrac{d}{dct}\left(\tfrac{1}{R_v} + \tfrac{\dot{R}}{R}\right) , \qquad (3.11)\\
&u_0 = -1 \quad , \quad u_r = 0.
\end{aligned}
$$

In order to compactly represent the dynamics of these equations, it is convenient to define the reduced cosmological scale $\mathcal{R}$ using

$$\frac{\dot{\mathcal{R}}}{\mathcal{R}} \equiv \frac{1}{R_v} + \frac{\dot{R}}{R}. \qquad (3.12)$$

Combining terms in Eq. 3.11, one obtains the first law of thermodynamics for an adiabatic expansion in terms of this scale

$$d(\rho\mathcal{R}^3) = -Pd\mathcal{R}^3. \qquad (3.13)$$

The dynamics of the reduced cosmological scale is directly determined by the energy content of the cosmology as expressed in Eq. 3.11. During epochs dominated by constant energy density, the dynamics is dominated by static values for R_v, with $\dot{R} \approx 0$, and $\mathcal{R} \simeq \mathcal{R}_v e^{ct/R_v}$. If there is an initial inflation followed by a long term adjustment towards dark energy domination, one expects the micro-physics to modify the dynamic content of the density in a manner that causes the scales to behave as follows:

$$
\begin{array}{cll}
R_I \Leftarrow R_v(ct) \Rightarrow R_\Lambda & for & 0 \leftarrow t/\tau_I \rightarrow \infty \\
\rho_I \Leftarrow \rho(ct) = \rho_v + \rho_{thermal} \Rightarrow \rho_\Lambda & for & 0 \leftarrow t/\tau_{I,c} \rightarrow \infty \\
0 \Leftarrow \dot{R}(ct) \Rightarrow 0 & for & 0 \leftarrow t/\tau_c \rightarrow \infty \\
R(0) \Leftarrow R(ct) \Rightarrow R_c & for & 0 \leftarrow t/\tau_c \rightarrow \infty
\end{array} \qquad (3.14)
$$

where the cosmological time scale is expected to be orders of magnitude greater than the inflationary time scale $\tau_c >> \tau_I$. It is non-trivial to develop a transformation that diagonalizes the metric in Eq. 3.9 as was done for the deSitter geometry in Eq. 3.6, unless the scales correspond in late times $R_c = R_\Lambda$. During the intermediate epoch $\tau_I < t < \tau_c$, the dynamics of the reduced scale $\mathcal{R}$ is essentially the same as that of the scale R, driven by the thermal energy content.

The dynamics can be expressed solely in terms of the energy content. Using Eq. 3.11

$$\frac{d}{dct}\rho = -\sqrt{\frac{24\pi G_N \rho}{c^4}}(P+\rho). \tag{3.15}$$

The usual form for the equation of state of the thermal content will be assumed:

$$\begin{aligned} \rho &= \rho_v + \rho_{thermal} \\ P &= P_v + P_{thermal} = -\rho_v + w\rho_{thermal}, \end{aligned} \tag{3.16}$$

where $w = 1/3$ for radiation and $w = 0$ for pressureless matter. Thus, Eq. 3.15 can be re-expressed

$$\frac{d}{dct}\rho_{thermal} = -\sqrt{\frac{24\pi G_N}{c^4}(\rho_v + \rho_{thermal})}(1+w)\rho_{thermal}. \tag{3.17}$$

Druing the intermediate (thermal) period $\tau_I << t << \tau_c$, one assumes $\rho_v << \rho_{thermal}$ and $\left|\frac{\dot{R}}{R}\right| >> \frac{1}{R_v} \cong const$, the thermal density satisfies

$$\begin{aligned} &\tfrac{d}{dct}\rho_{thermal} = -\sqrt{\tfrac{24\pi G_N}{c^4}}(1+w_*)\rho_{thermal}^{3/2} \\ &\left(\tfrac{\rho_*}{\rho_{thermal}}\right)^{1/2} \cong 1 + \sqrt{\tfrac{6\pi G_N \rho_*}{c^4}}(1+w_*)(ct - ct_*), \end{aligned} \tag{3.18}$$

which are the same as the behaviors predicted by the Friedman-Lemaitre equations during the thermal period[3].

During the very early $t << \tau_I$ and very late $t > \tau_c$ epochs, the evolution of the thermal component of density can likewise be determined:

$$\begin{aligned} &\tfrac{d}{dct}\rho_{thermal} \approx -\sqrt{\tfrac{24\pi G_N \rho_v}{c^4}}(1+w_*)\rho_{thermal} \\ &log\left(\tfrac{\rho_{thermal}}{\rho_*}\right) \approx -\tfrac{3}{R_v}(1+w_*)(ct - ct_*), \end{aligned} \tag{3.19}$$

clearly indicating exponential decrease in the thermal density during these epochs.

As an aside, one can examine the growth of measurable dark energy generated as vacuum energy during microscopic thermalization of collective macroscopic motions. If the dark energy is of the form

$$\rho_{dark} \sim \sum_{modes} \frac{1}{2}\hbar k v_p \sim (k_{UV}^4 - k_{IR}^4(t))\hbar v_p, \tag{3.20}$$

the measurable infrared modes are expected to scale inversely with the horizon, with essentially luminal speed. Thus, the measurable dark energy grows like

$$\begin{aligned} \dot{\rho}_{dark} &\sim \frac{\rho_{darkUV}}{t_{UV}} \left(\frac{t_{UV}}{t}\right)^4 \\ \rho_{dark} &\sim \rho_\Lambda \left(1 - \left(\frac{t_{UV}}{t}\right)^3\right). \end{aligned} \tag{3.21}$$

Curvature arguments

As a final question, the inclusion of curvature will be briefly explored. If curvature is introduced by the inclusion in the FRW part of the metric Eq. 3.9 in the form

$$\begin{aligned} g_{\mu\nu} = &-\left(1 - \frac{R^2(ct)r^2}{R_v^2(ct)}\right) c^2dt^2 - 2\frac{R^2(ct)r}{R_v(ct)} cdt\, dr \\ &+R^2(ct)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right), \end{aligned} \tag{3.22}$$

an "open" cosmology ($\kappa = +1$) is excluded by the consistency condition Eq. 2.3 with $(u_r)^2 < 0$, whereas a "closed" cosmology ($\kappa = -1$) must be finely tuned, as similarly can be infered from the FRW cosmology[4]. Therefore, fluid consistency constraints exclude a nonvanishing value for κ as a likely scenario.

4 Conclusions

A form for a metric that incorporates evolution from an early inflationary epoch through a thermal period towards a final deSitter state has been developed. The metric can be used to explore the transition state associated with the initial thermalization period with as close a correspondence with the FRW geometry as desired by appropriate choice of the relative cosmological scales. If an initial inflation state is confirmed, it is hoped that future work utilizing this metric will give insight into the microscopic contributions to the evolution of cosmological dark energy.

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